18th IFAC World Congress
Convex Optimization in Robust Control

Wednesday, September 2, 2011
17.00-17.20 Room “Mons. Colombo”

FAST: An Algorithm for the Scenario Approach with Reduced Sample Complexity

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Outline

"FAST - An Algorithm for the Scenario Approach with Reduced Sample Complexity"

I. The Scenario Approach

II. An issue

III. FAST gets around the issue

Background Information

"FAST -
An Algorithm for the Scenario Approach
with Reduced Sample Complexity"

I. The Scenario Approach

An issue

FAST gets around the issue

A convex problem...

$$\min_\gamma \ell(\gamma)$$

A convex problem...

$$\min_\gamma \ell(\gamma)$$

$$\gamma \in \Gamma \subseteq \mathbb{R}^d$$

A convex problem...

$$\min_{\gamma} \ell(\gamma)$$

$$\gamma \in \Gamma \subseteq \mathbb{R}^d$$

$$\ell(\gamma) : \Gamma \to \mathbb{R}$$ CONVEX

...as a design problem

$$\min_\gamma \ell(\gamma)$$

$$\gamma \in \Gamma \subseteq \mathbb{R}^d$$

controller parameters

$$\ell(\gamma) : \Gamma \to \mathbb{R}$$

CONVEX

performance (cost)

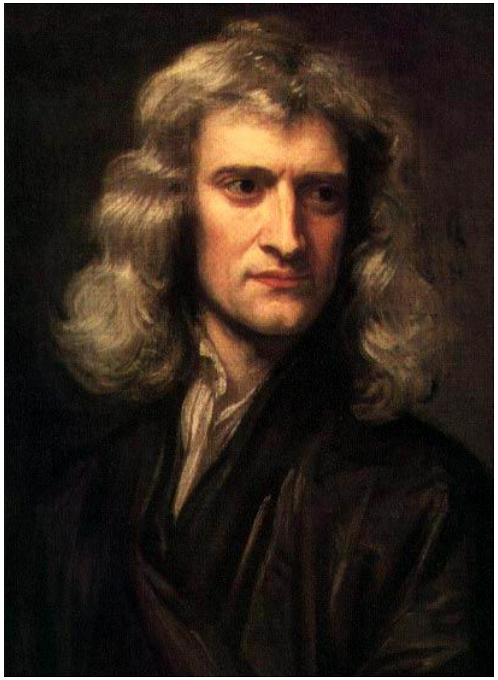
In real life...

$$\min_\gamma \ell_\delta(\gamma)$$

Uncertain parameter

The Scenario Approach: a motto

Let reality speak for itself



"*Hypotheses non fingo*"
Sir Isaac Newton

Just look at data!

Collect observations ("scenarios"):

$$\delta^{(1)}, \delta^{(2)}, \ldots, \delta^{(N)}$$

The Scenario Program

Collect observations ("scenarios"):

$$\delta^{(1)}, \delta^{(2)}, \ldots, \delta^{(N)}$$

and solve the "scenario program":

The Scenario Program

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$$\delta^{(1)}, \delta^{(2)}, \ldots, \delta^{(N)}$$

and solve the "scenario program":

$$\ell_{\delta^{(1)}}(\gamma), \ell_{\delta^{(2)}}(\gamma), \ldots, \ell_{\delta^{(N)}}(\gamma)$$

The Scenario Program

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$$\delta^{(1)}, \delta^{(2)}, \ldots, \delta^{(N)}$$

and solve the "scenario program":

$$\ell_{\delta^{(1)}}(\gamma), \ell_{\delta^{(2)}}(\gamma), \ldots, \ell_{\delta^{(N)}}(\gamma)$$

$$\max_{i=1,\ldots,N} \ell_{\delta^{(i)}}(\gamma)$$

The Scenario Program

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and solve the "scenario program":

$$\ell_{\delta^{(1)}}(\gamma), \ell_{\delta^{(2)}}(\gamma), \dots, \ell_{\delta^{(N)}}(\gamma)$$

$$\min_{\gamma} \max_{i=1,\dots,N} \ell_{\delta^{(i)}}(\gamma)$$

The Scenario Program

Collect observations ("scenarios"):

$$\delta^{(1)}, \delta^{(2)}, \dots, \delta^{(N)}$$

and solve the "scenario program":

$$\ell_{\delta^{(1)}}(\gamma), \ell_{\delta^{(2)}}(\gamma), \dots, \ell_{\delta^{(N)}}(\gamma)$$

$$\min_{\gamma} \max_{i=1,\dots,N} \ell_{\delta^{(i)}}(\gamma) \qquad \gamma^*$$

worst-case design parameter

An equivalent reformulation

$$\min_{\gamma} \max_{i=1,\ldots,N} \ell_{\delta^{(i)}}(\gamma)$$

⇓

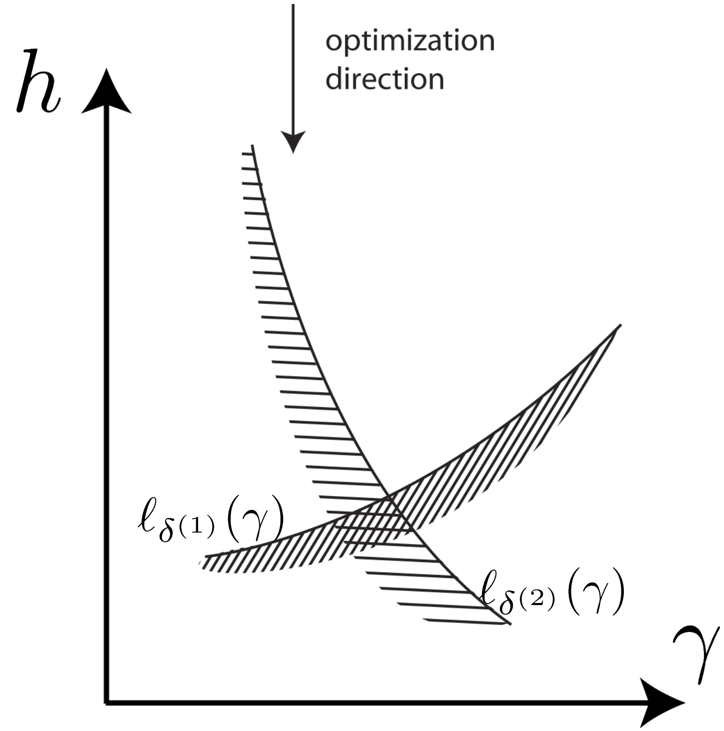
$$\min_{\gamma, h} h$$

$$\text{subject to: } \begin{aligned} \ell_{\delta^{(1)}}(\gamma) &\leq h, \\ \ell_{\delta^{(2)}}(\gamma) &\leq h, \\ &\vdots \\ \ell_{\delta^{(N)}}(\gamma) &\leq h. \end{aligned}$$

The Scenario Program

$$\min_{\gamma,h} h$$

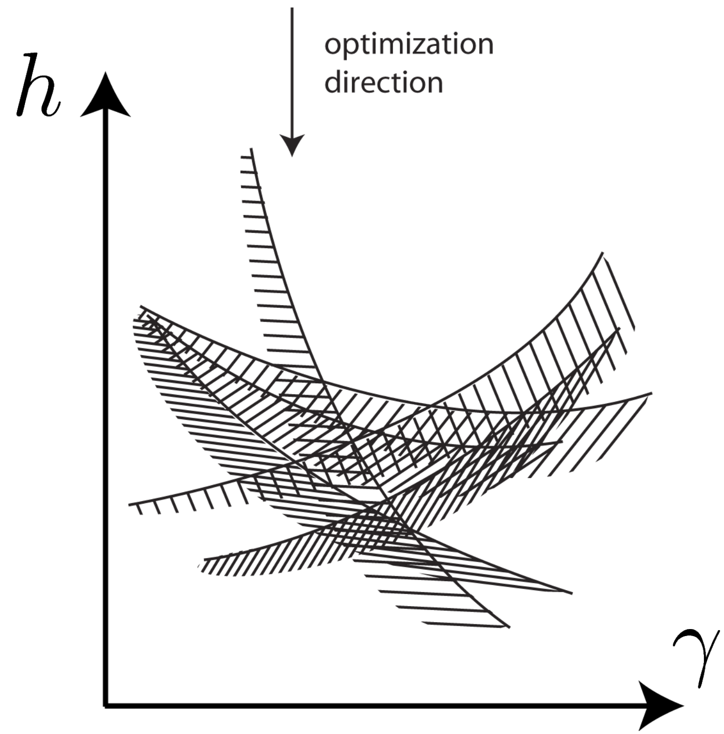
$$\text{subject to: } \ell_{\delta^{(1)}}(\gamma) \le h, \quad \ell_{\delta^{(2)}}(\gamma) \le h,$$



The Scenario Program

$$\min_{\gamma, h} h$$

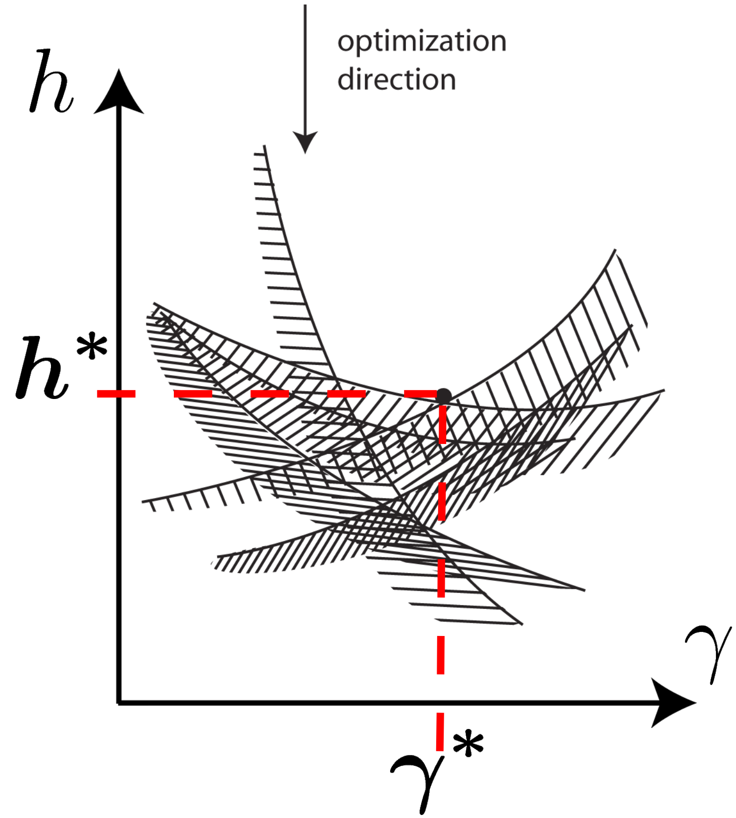
$$\text{subject to: } \ell_{\delta^{(1)}}(\gamma) \leq h,$$
$$\ell_{\delta^{(2)}}(\gamma) \leq h,$$
$$\vdots$$
$$\ell_{\delta^{(N)}}(\gamma) \leq h.$$



The Scenario Solution

$$\min_{\gamma, h} h$$

$$\text{subject to: } \ell_{\delta^{(1)}}(\gamma) \leq h, \quad \ell_{\delta^{(2)}}(\gamma) \leq h, \quad \vdots \quad \ell_{\delta^{(N)}}(\gamma) \leq h.$$



$$\text{scenario solution: } (\gamma^*, h^*)$$

$$h^* = \max_{i=1,\ldots,N} \ell_{\delta^{(i)}}(\gamma^*)$$

Recipe (by Campi, Garatti, Calafiore)

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$$N := \text{function}(\epsilon, d)$$

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$$N := \text{function}(\epsilon, d)$$

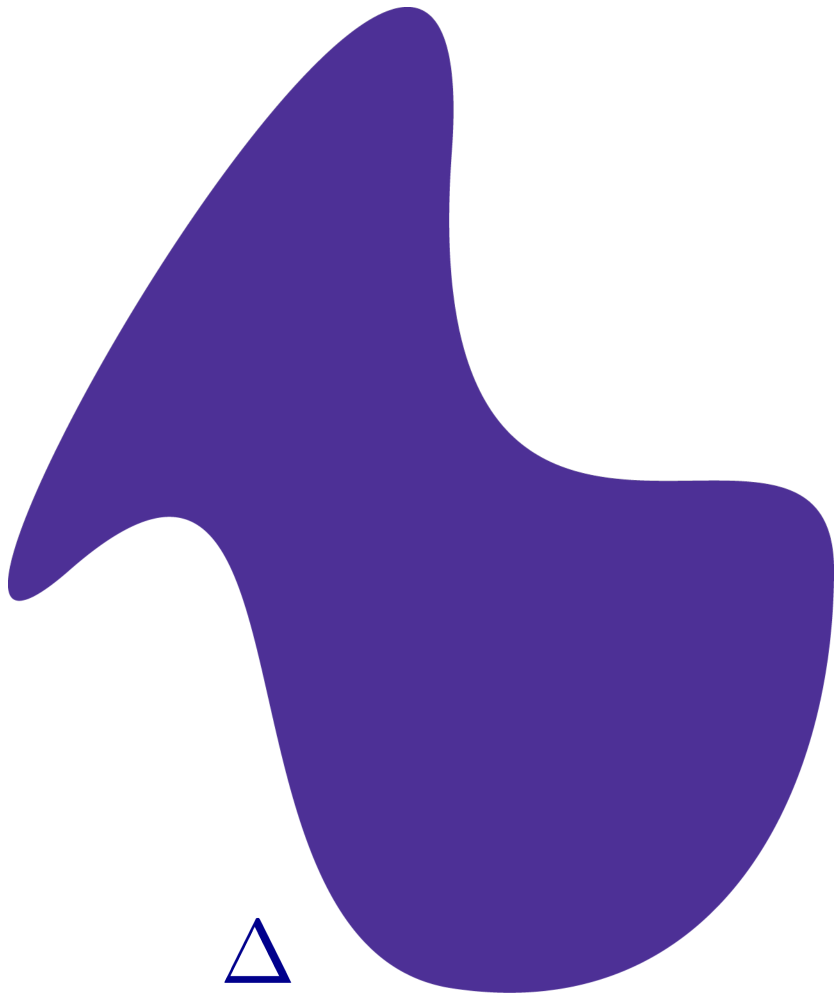
Recipe (by Campi, Garatti, Calafiore)

$$\text{If } N := \text{function}(\epsilon, d)$$

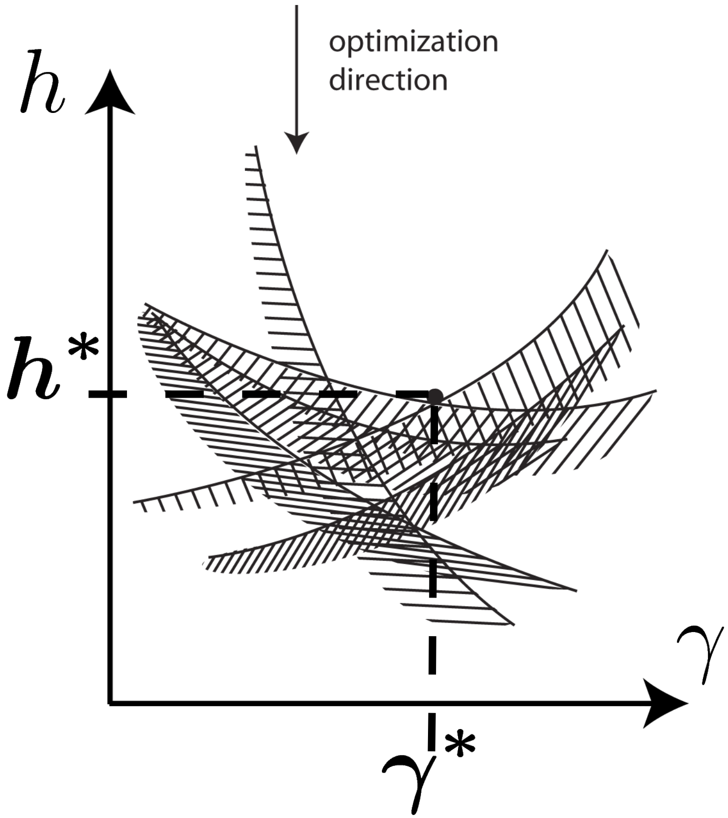
then

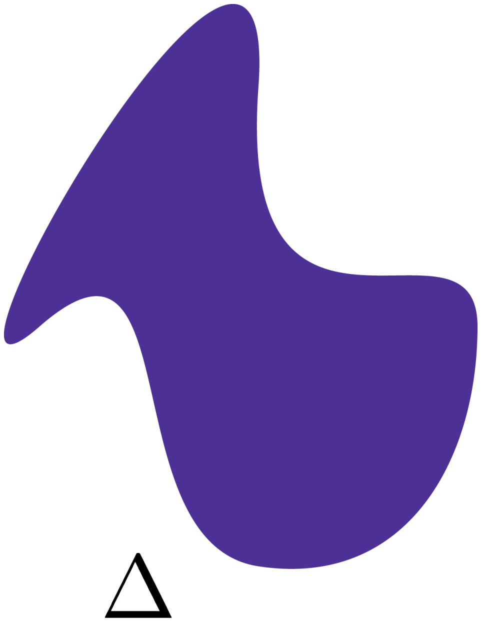
the scenario solution
has the desired (ϵ) generalization properties

Uncertainty Domain

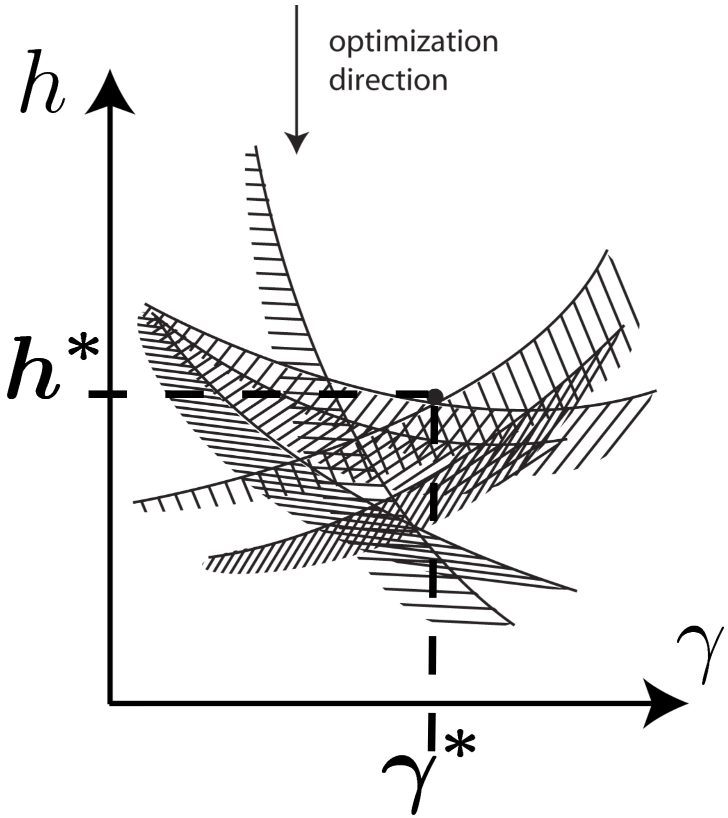


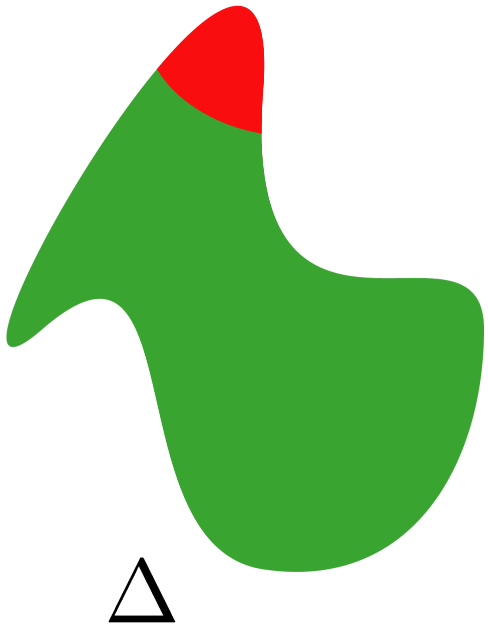
Uncertainty Domain



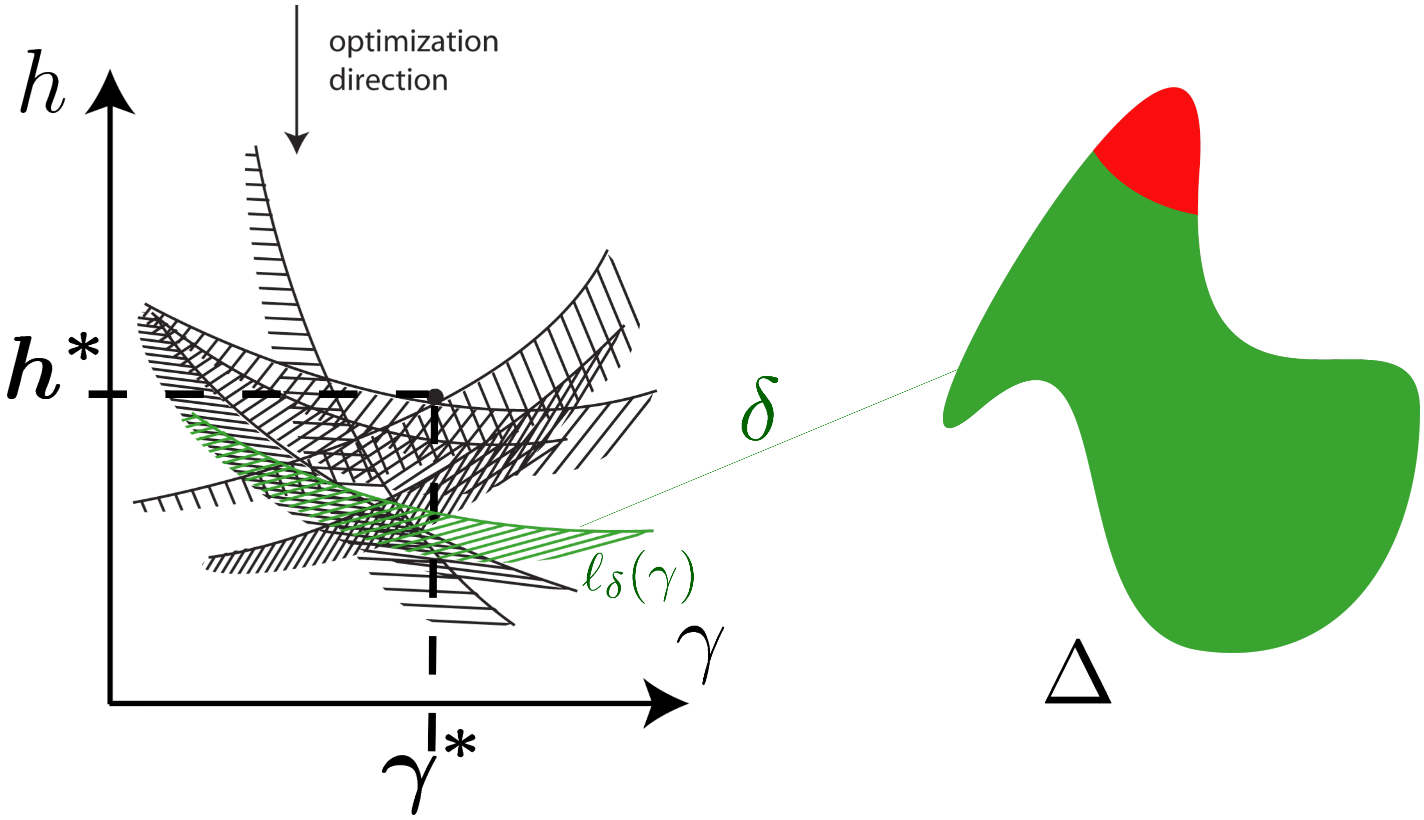


Uncertainty Domain

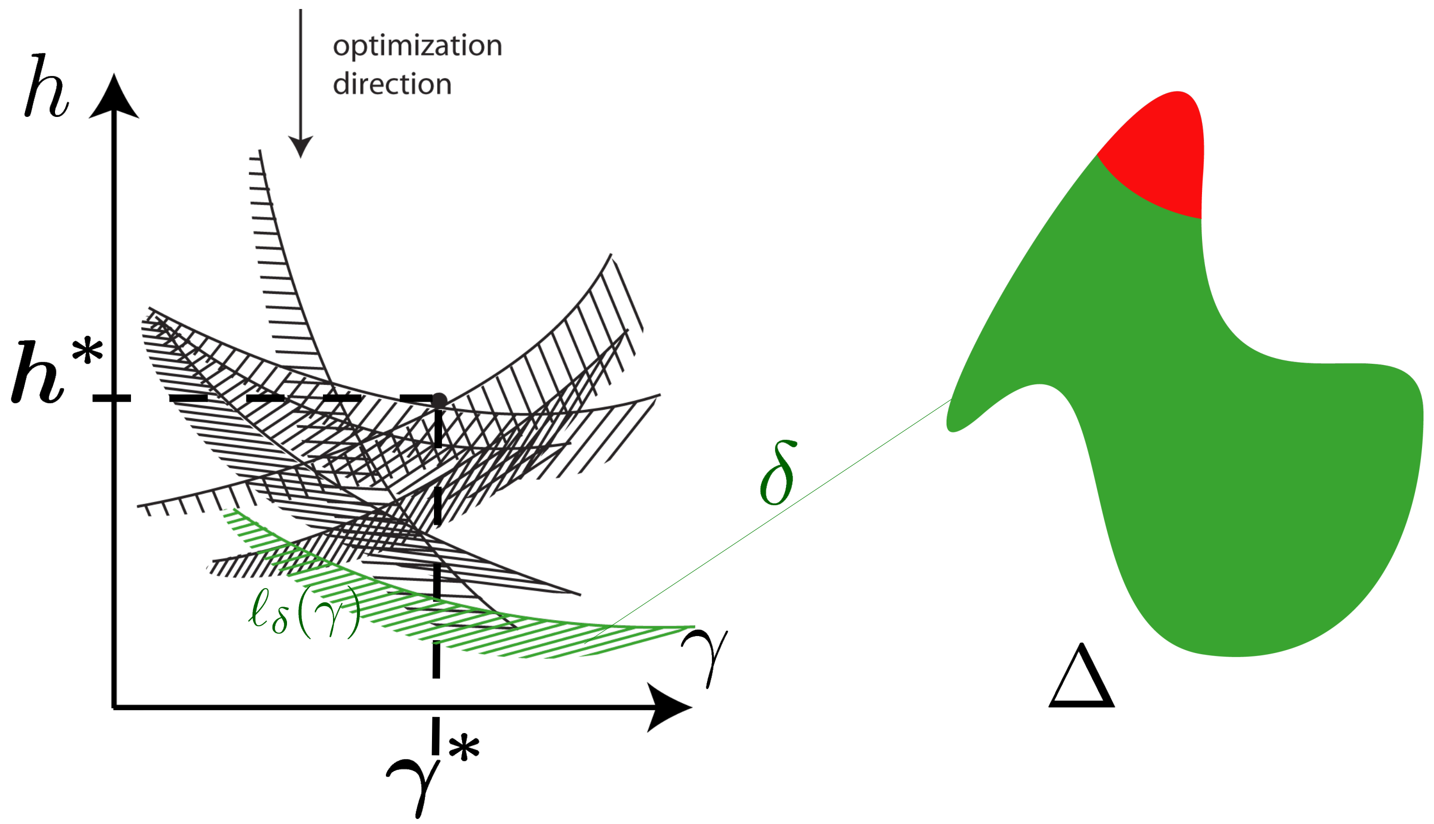




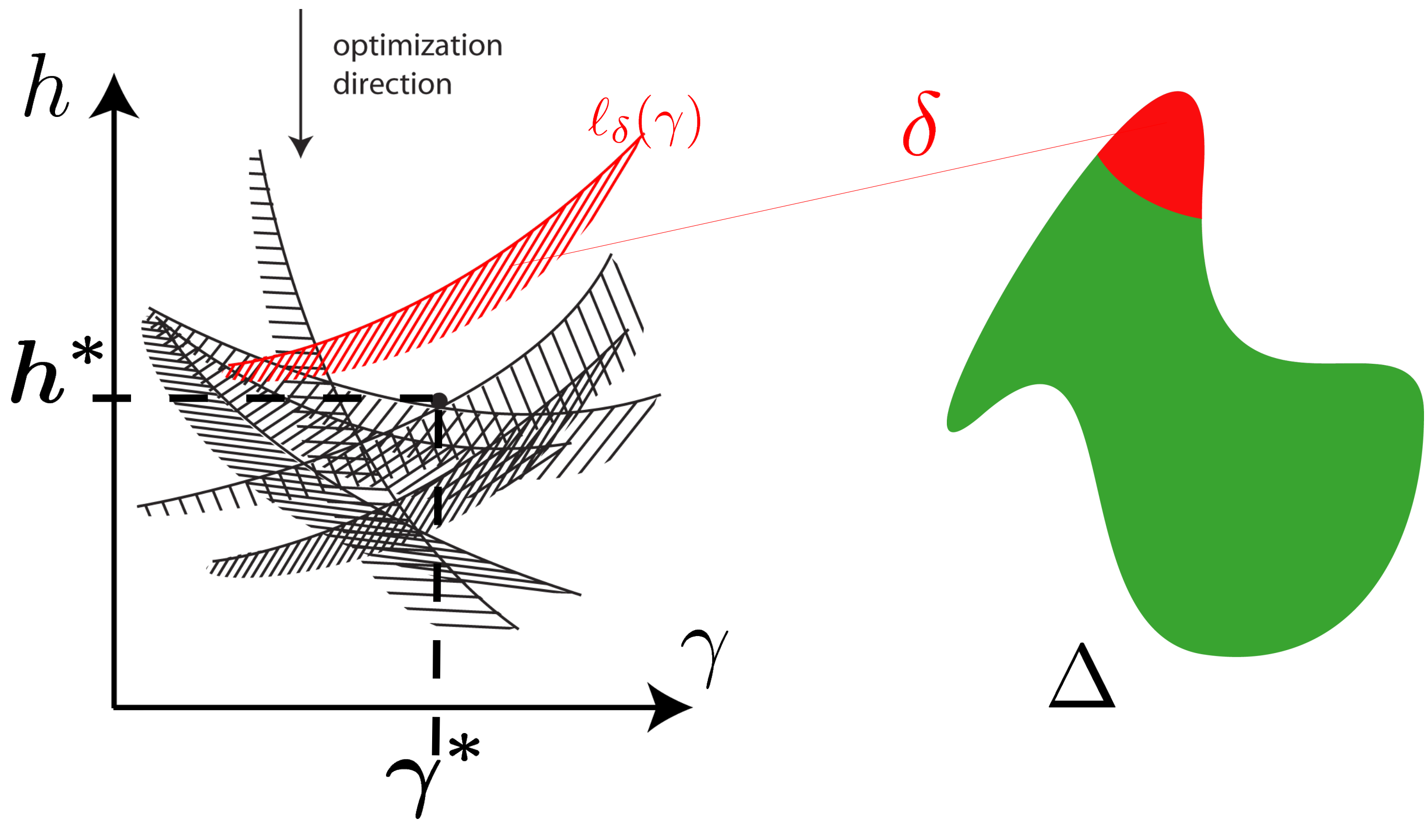
Uncertainty Domain



Uncertainty Domain

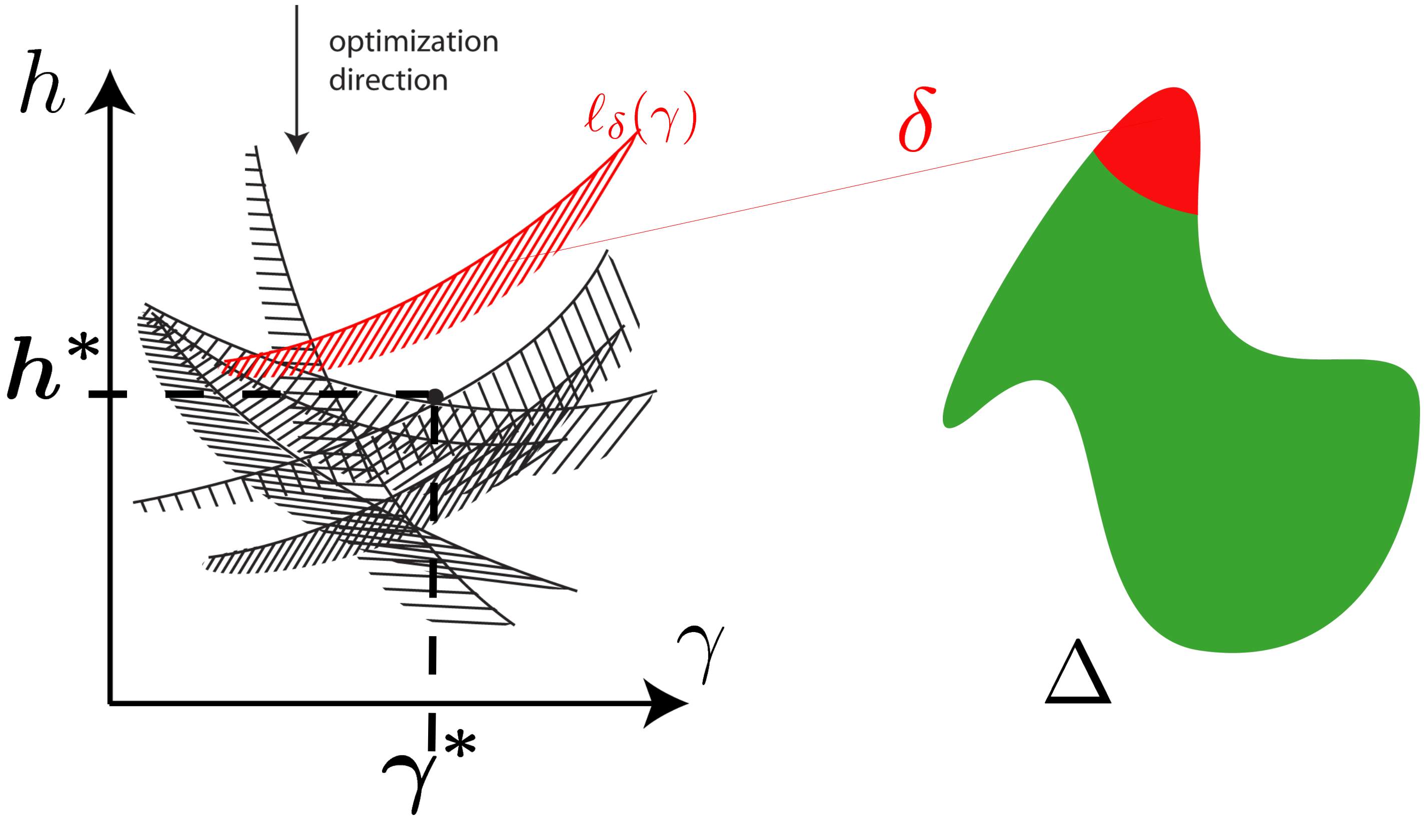


Violation



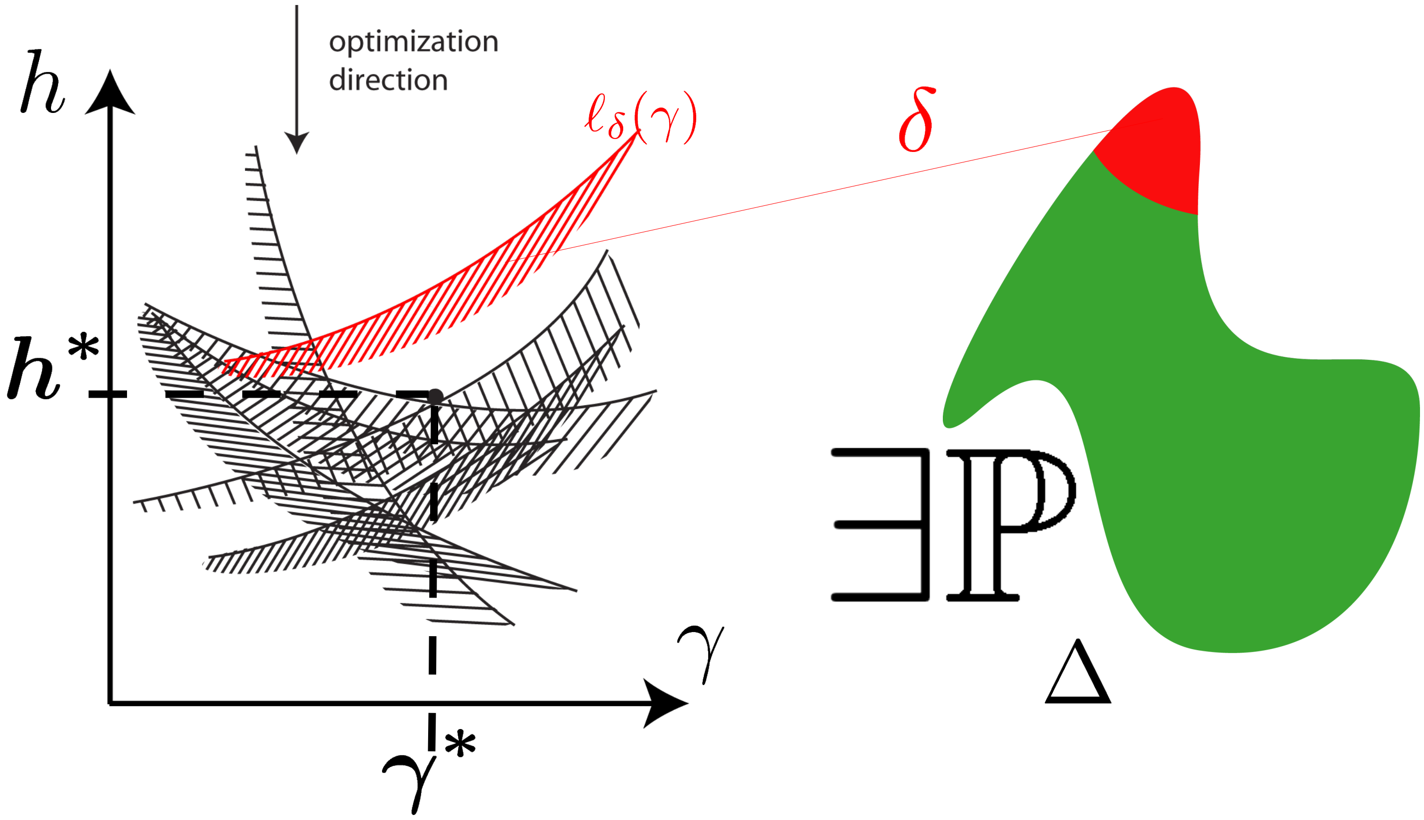
optimization direction
h
$\ell_\delta(\gamma)$
δ
h^*
γ
γ^*
Δ

Violation



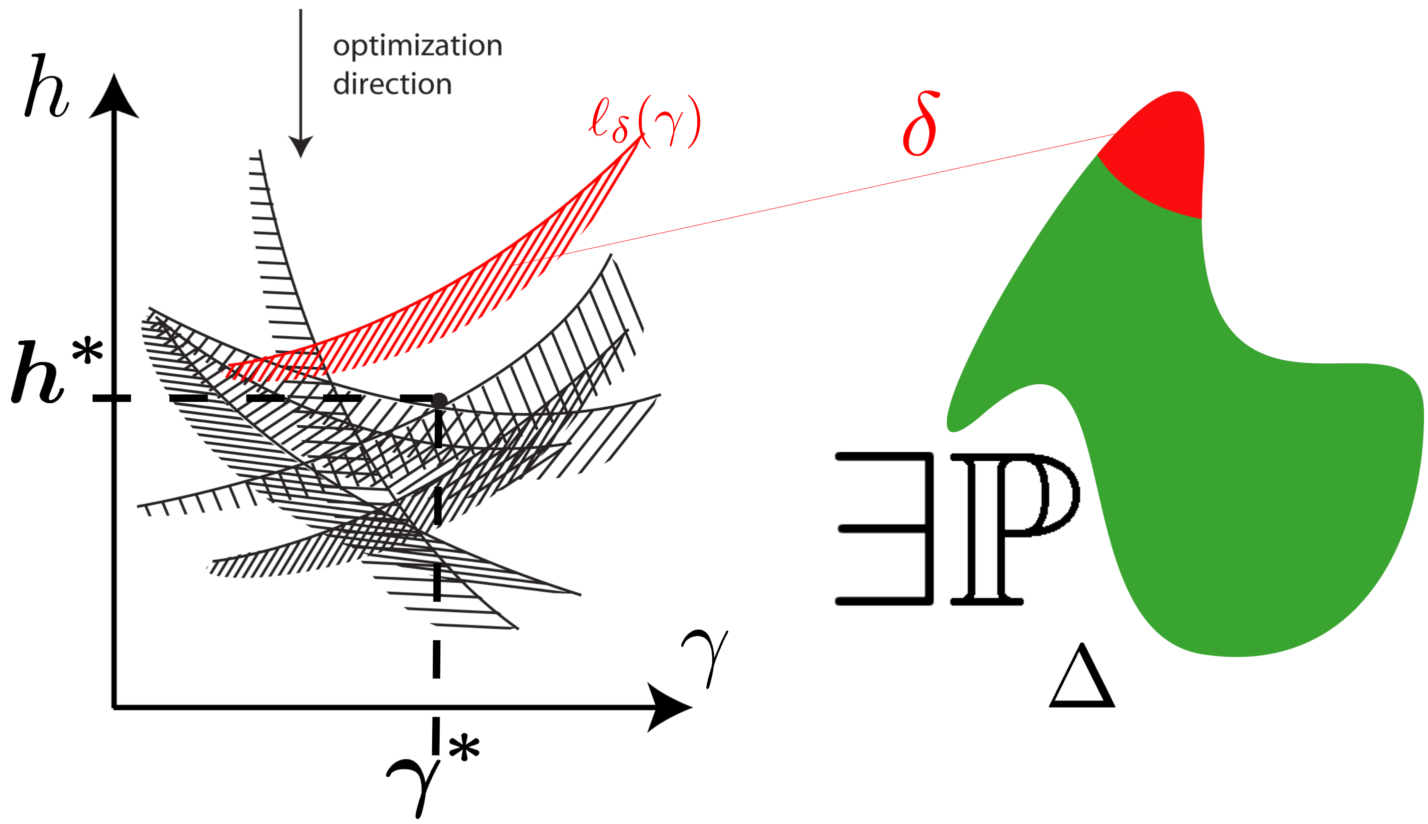
$$\ell_{\delta}(\gamma^*) > h^*$$

Assumption



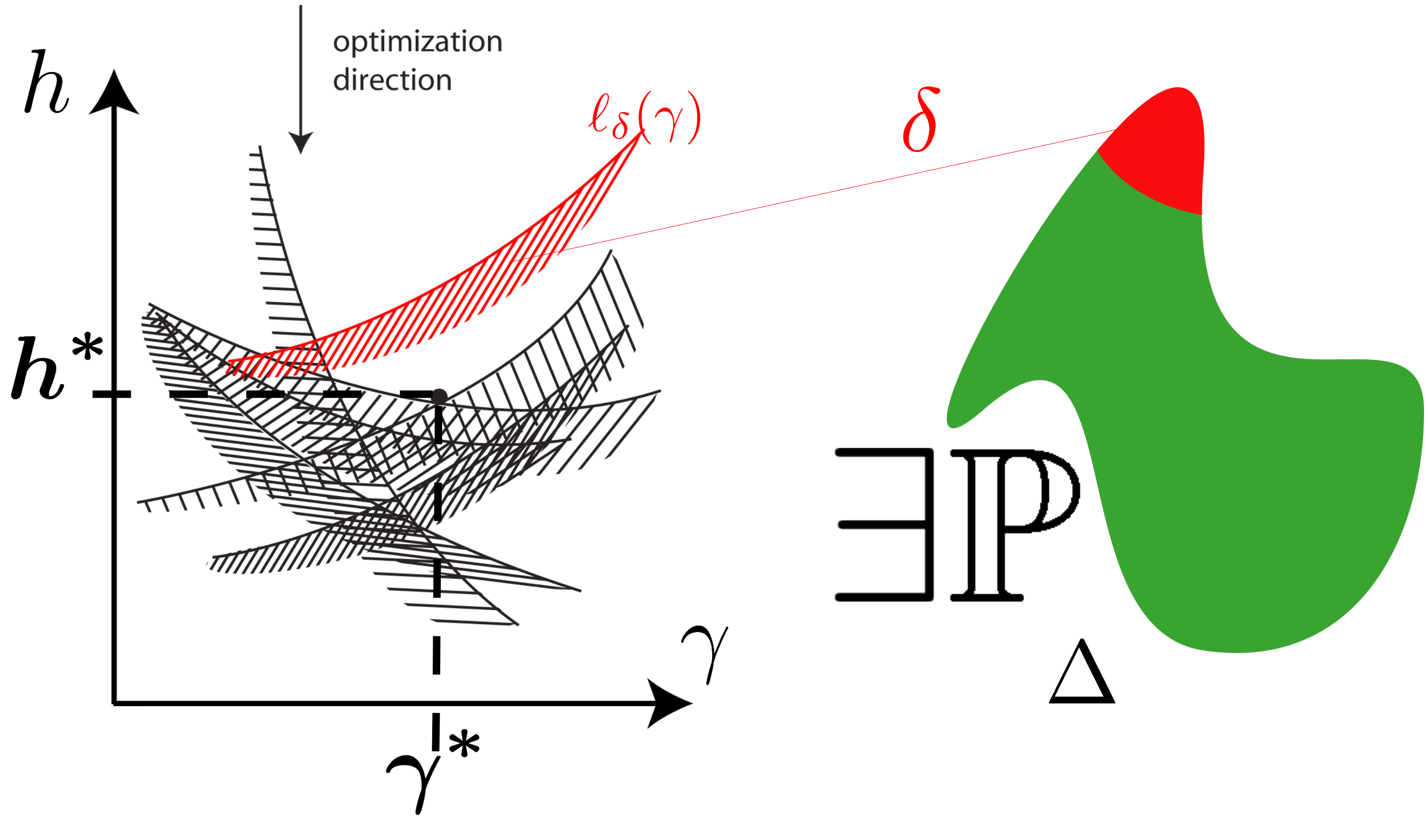
$$\ell_\delta(\gamma^*) > h^*$$

Guarantees!



$$\underbrace{\mathbb{P}\{\ell_\delta(\gamma^*) > h^*\}}_{\text{Red set probability}} \leq \epsilon$$

Guarantees!



$$\underbrace{\mathbb{P}\{\ell_\delta(\gamma^*) > h^*\}}_{\text{Violation probability}} \le \epsilon$$

Take-home message #1

$$\text{If } N := \text{function}(\epsilon, d)$$

then

$$\underbrace{\mathbb{P}\{\ell_\delta(\gamma^*) > h^*\}}_{\text{Violation probability}} \leq \epsilon$$

Sample Complexity

Sample-complexity

$$\text{If } N := \text{function}(\epsilon, d)$$

then

$$\mathbb{P}\{\ell_\delta(\gamma^*) > h^*\} \leq \epsilon$$

Explicit formula (Calafiore, Alamo, Tempo, Luque)

$$\text{If } N := 2\frac{d}{\epsilon} + \frac{44}{\epsilon}$$

then[†]

$$\mathbb{P}\{\ell_\delta(\gamma^*) > h^*\} \leq \epsilon$$

[†](modulo a confidence parameter...)

An issue

"FAST -
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with Reduced Sample Complexity"

The Scenario Approach

II. An issue

FAST gets around the issue

Bad news

$$\text{If } N := 2\frac{d}{\epsilon} + \frac{44}{\epsilon}$$

then

$$\mathbb{P}\{\ell_\delta(\gamma^*) > h^*\} \leq \epsilon$$

Nemirovski and Shapiro (2006), Oishi (2007)
– “... **it is difficult to apply this approach** with too many constraints ...”

The guilty term

$$\text{If } N := 2\frac{d}{\epsilon} + \frac{44}{\epsilon}$$

then

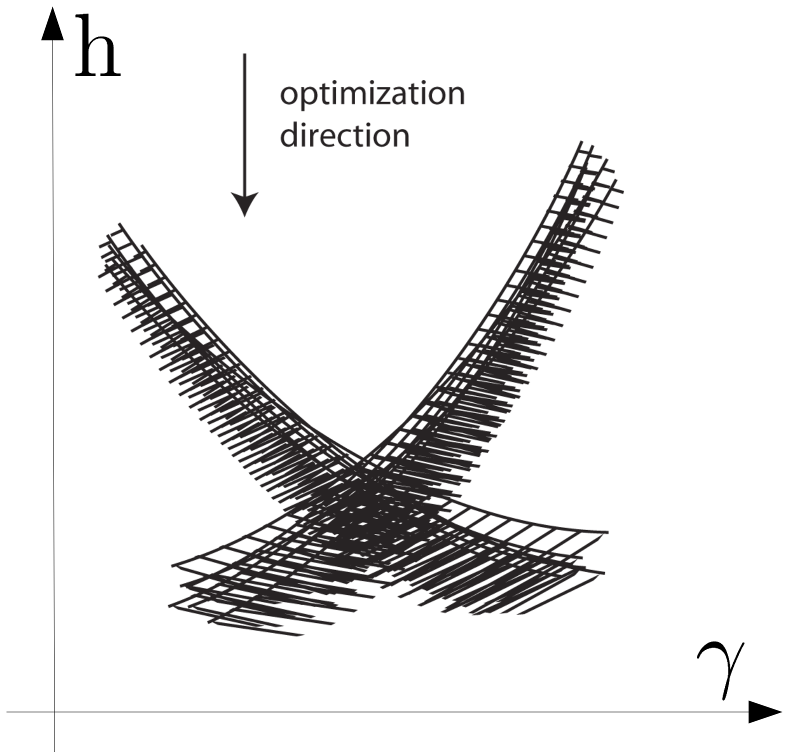
$$\mathbb{P}\{\ell_\delta(\gamma^*) > h^*\} \leq \epsilon$$

Nemirovski and Shapiro (2006), Oishi (2007)
– "... **it is difficult to apply this approach** with too many constraints ..."

Remark

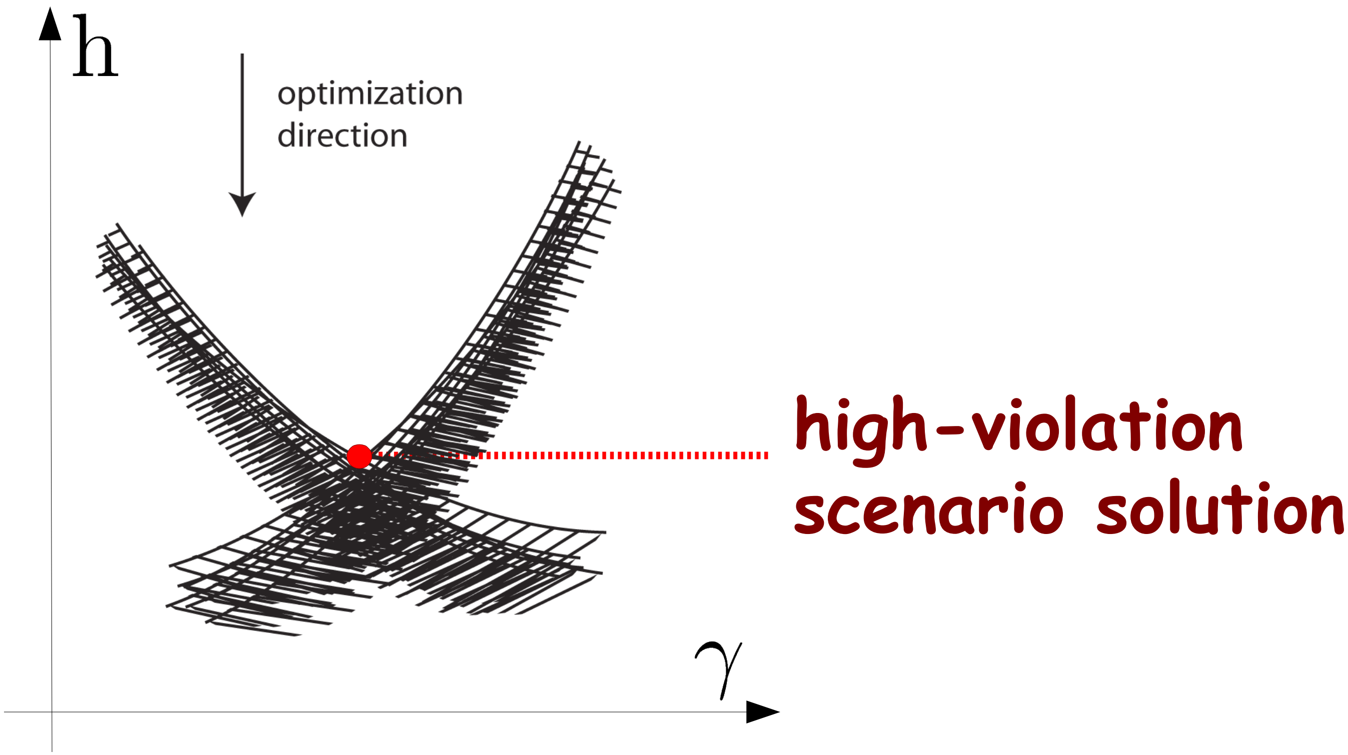
Campi and Garatti (2008): "this result ***is tight and cannot be improved***".

Too few scenarios



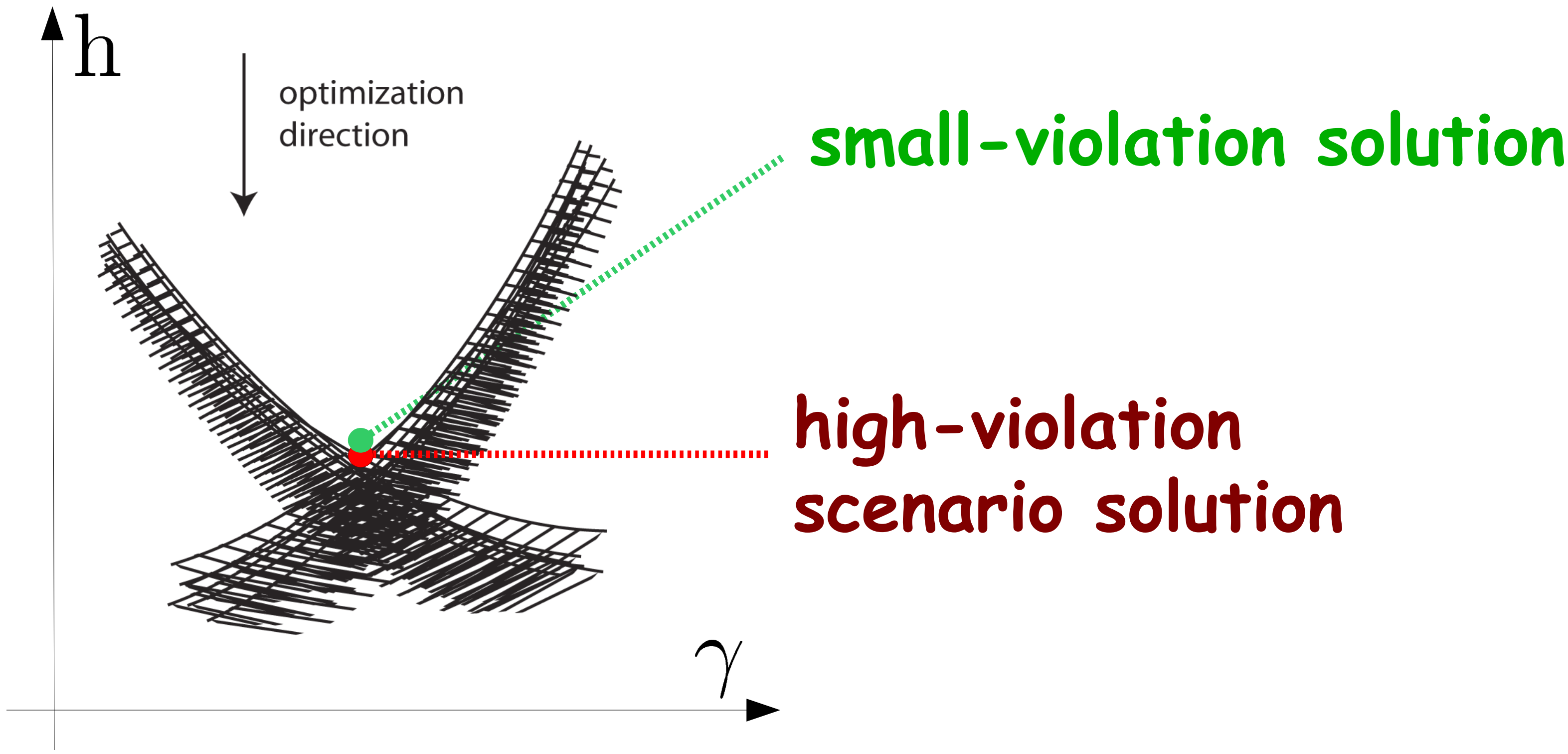
Optimization space with *too few* scenarios

Too few scenarios



Optimization space with *too few* scenarios

Move just a little bit from it...



Optimization space with *too few scenarios*

Third Part

“FAST -
An Algorithm for the Scenario Approach
with Reduced Sample Complexity”

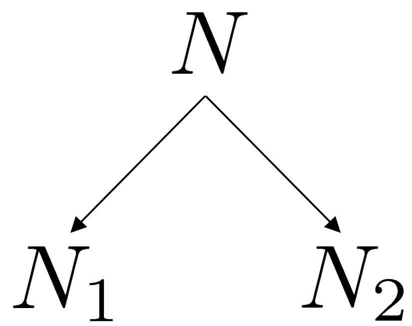
The Scenario Approach

An Issue

III. FAST gets around the issue

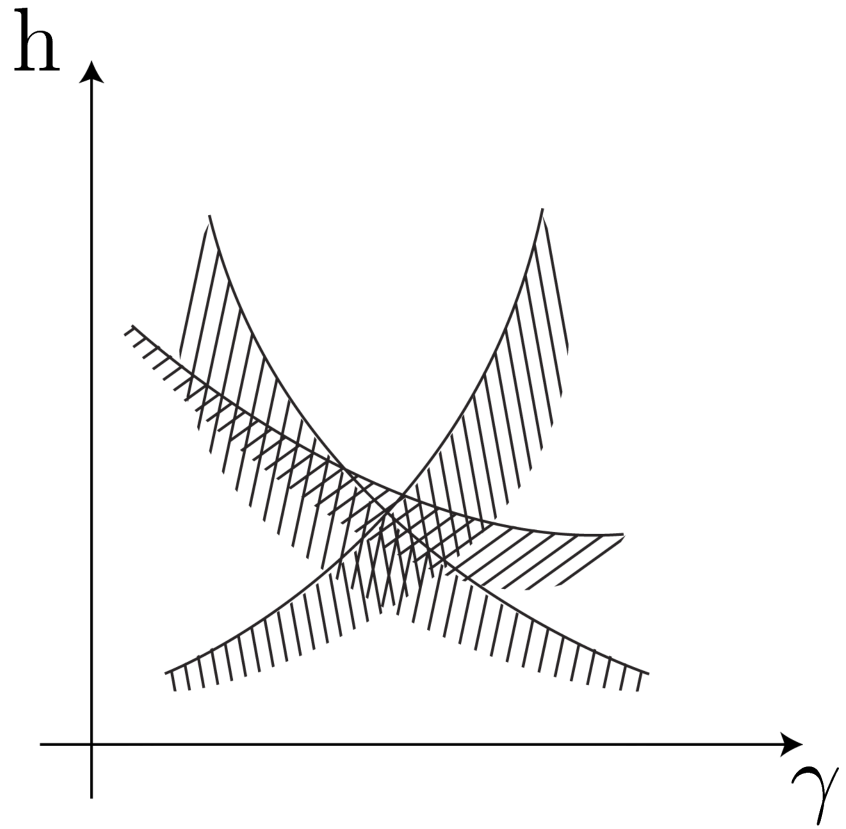
FAST Algorithm...

Fast **A**lgorithm for the **S**cenario **T**echnique



Step I

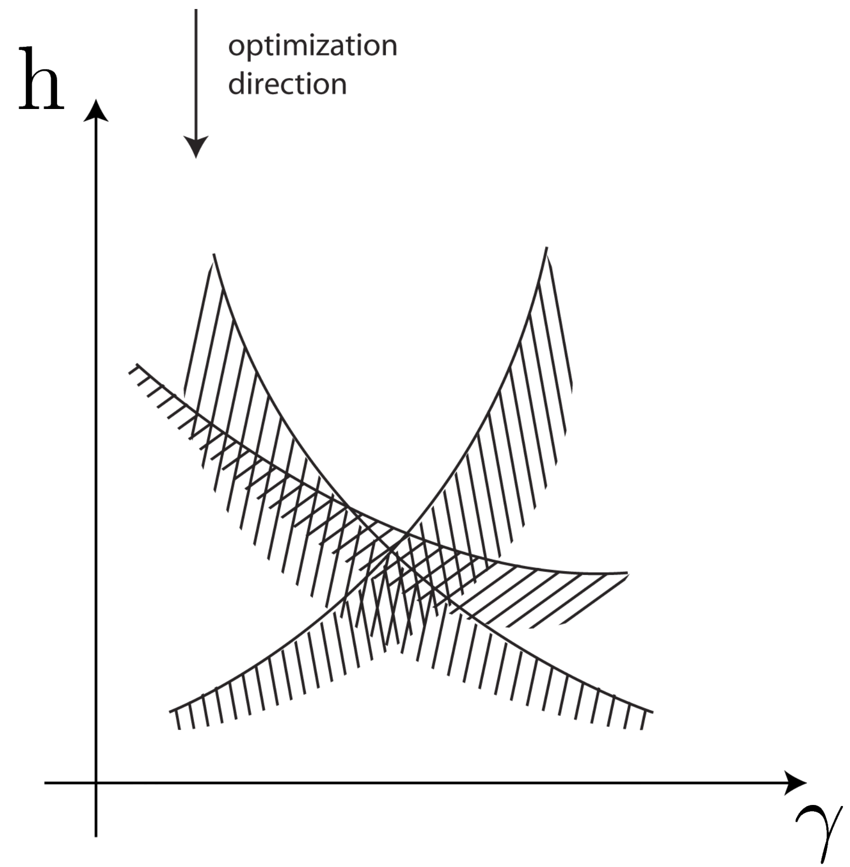
$$\delta^{(1)}, \ldots, \delta^{(N_1)}$$



Step I

$$\delta^{(1)}, \ldots, \delta^{(N_1)}$$

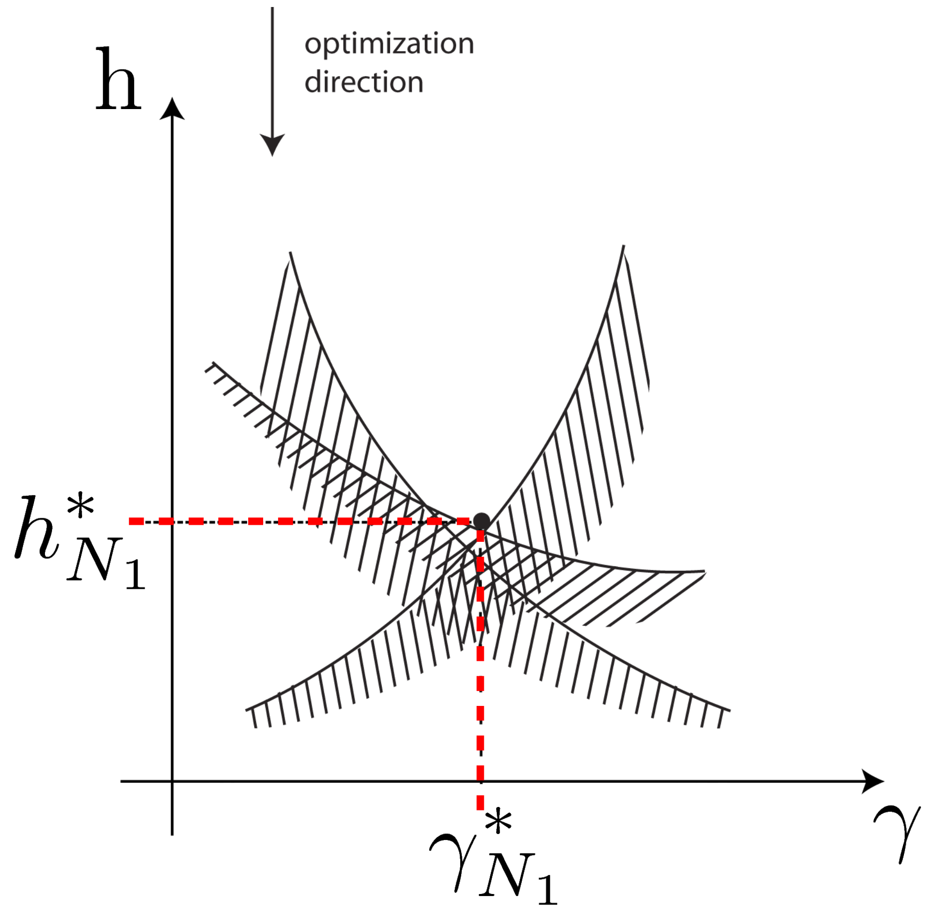
$$\begin{aligned} &\min_{\gamma,h} h \\ &\text{subject to: } \ell(\gamma)_{\delta^{(1)}} \leq h, \\ &\qquad\qquad\quad \ell(\gamma)_{\delta^{(2)}} \leq h, \\ &\qquad\qquad\qquad \vdots \\ &\qquad\qquad\quad \ell(\gamma)_{\delta^{(N_1)}} \leq h, \end{aligned}$$



Step I

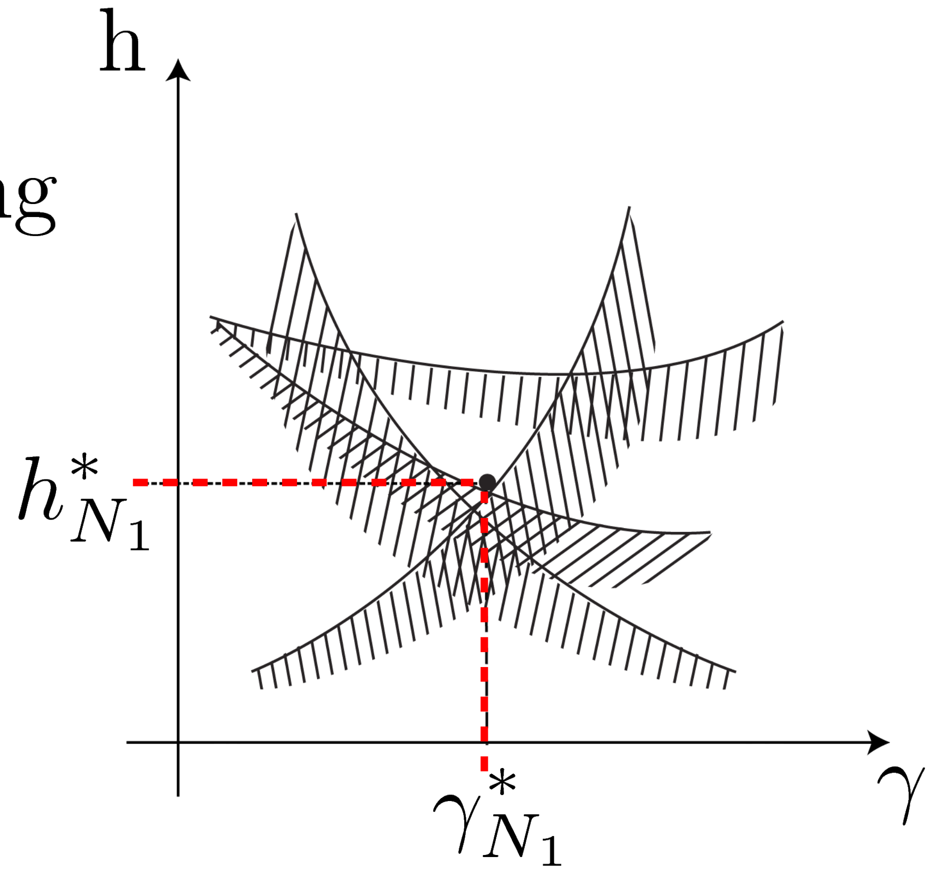
$$\delta^{(1)}, \ldots, \delta^{(N_1)}$$

$$\begin{aligned} \min_{\gamma, h} \; & h \\ \text{subject to: } & \ell(\gamma)_{\delta^{(1)}} \leq h, \\ & \ell(\gamma)_{\delta^{(2)}} \leq h, \\ & \vdots \\ & \ell(\gamma)_{\delta^{(N_1)}} \leq h, \end{aligned}$$



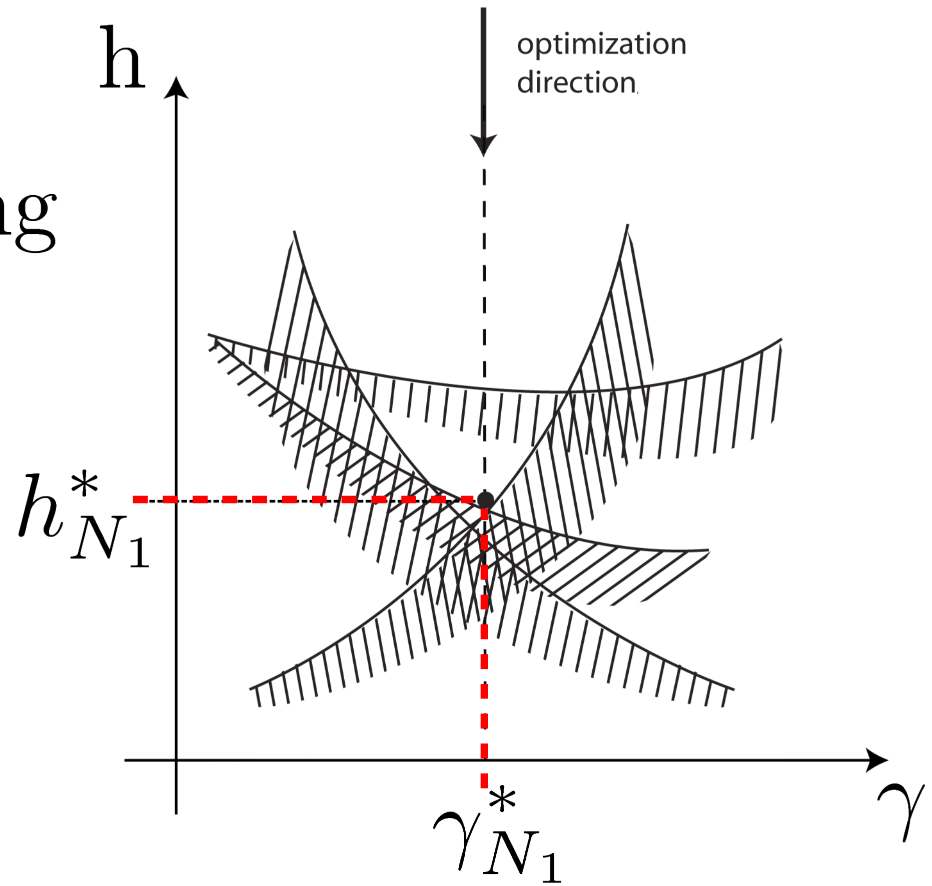
Step II

Consider the remaining N_2 scenarios...



Step II

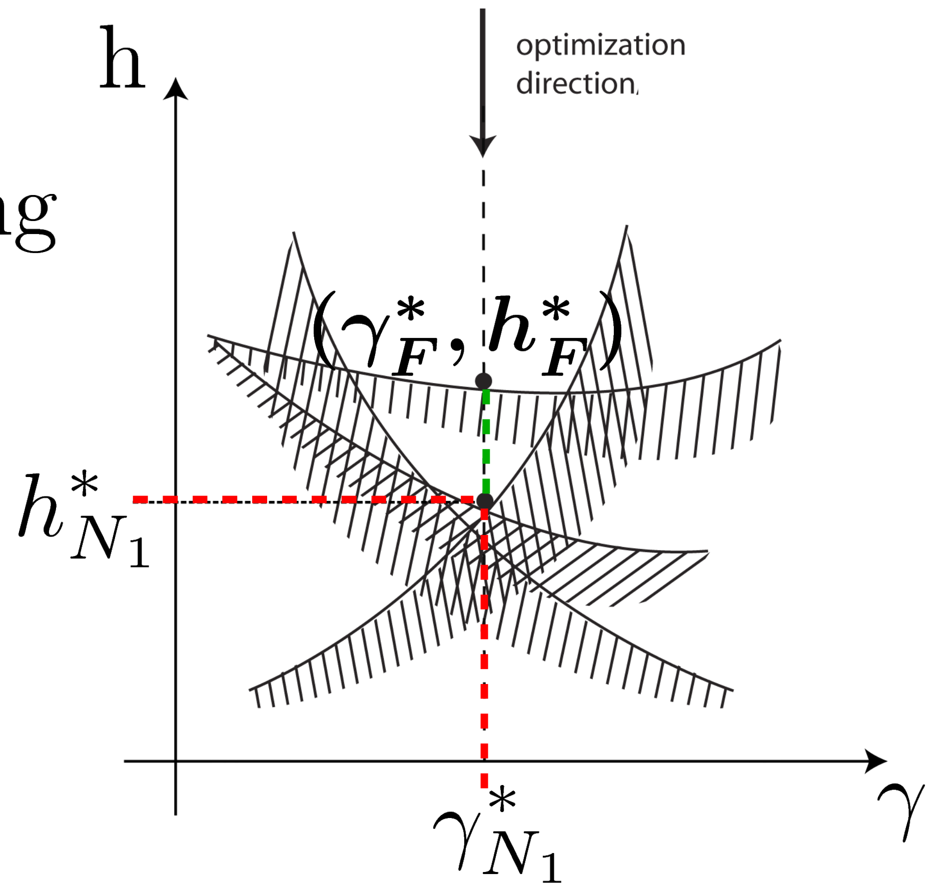
Consider the remaining N_2 scenarios...



FAST Solution

Consider the remaining N_2 scenarios...

1-dimensional LP



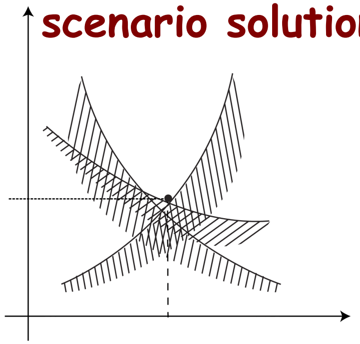
FAST algorithm

$\delta^{(1)}, \ldots, \delta^{(N_1)}$

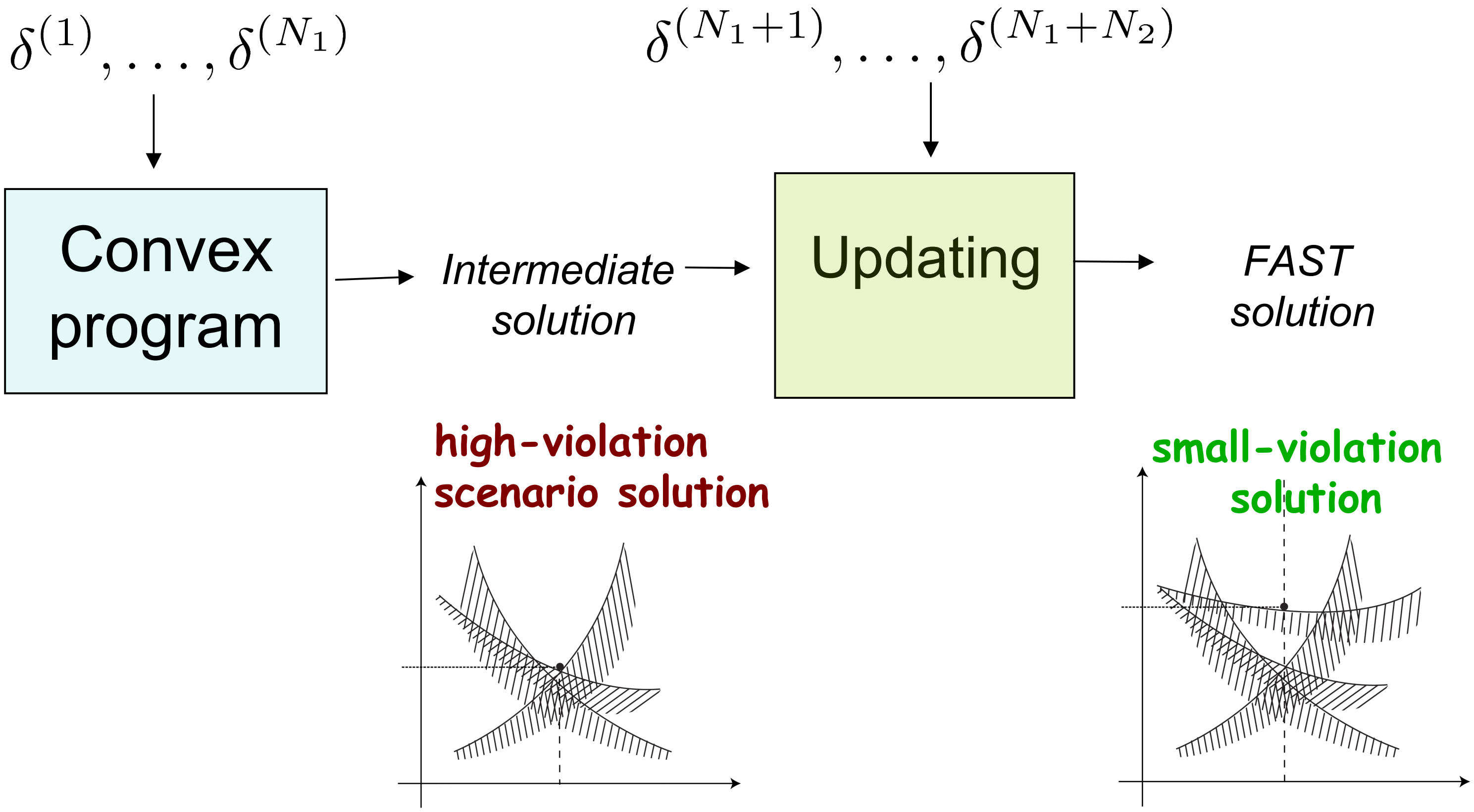
Convex program

Intermediate solution

high-violation scenario solution



FAST algorithm



FAST recipe (by Carè, Campi, Garatti)

If $N_1 := 20 \cdot d$

$N_2 := 21 \cdot \frac{1}{\epsilon}$

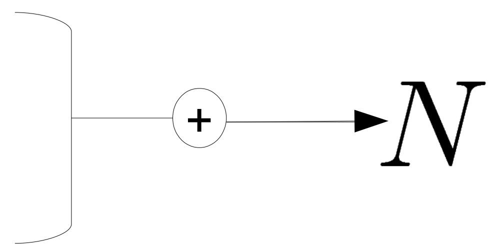
then

$$\mathbb{P}\{\ell_\delta(\gamma_F^*) > h_F^*\} \leq \epsilon$$

Additive dependence

If $N_1 := 20 \cdot d$

$N_2 := 21 \cdot \frac{1}{\epsilon}$



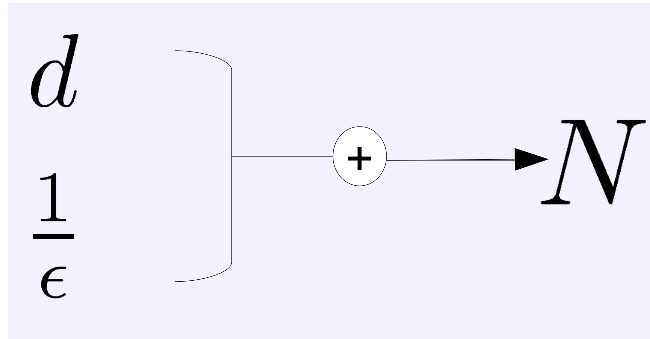
then

$$\mathbb{P}\{\ell_\delta(\gamma_F^*) > h_F^*\} \leq \epsilon$$

Additive dependence

$$\text{If} \quad N_1 := 20 \cdot d$$

$$N_2 := 21 \cdot \frac{1}{\epsilon}$$



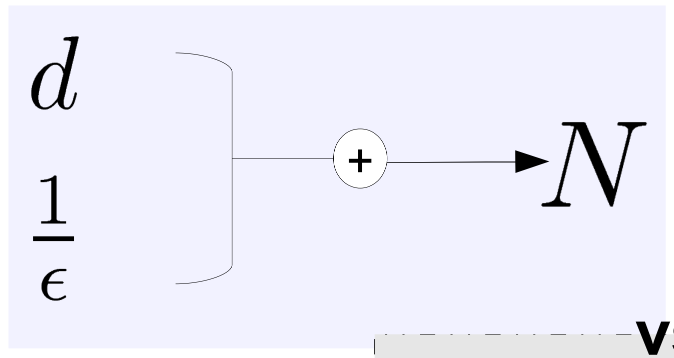
then

$$\mathbb{P}\{\ell_\delta(\gamma_F^*) > h_F^*\} \leq \epsilon$$

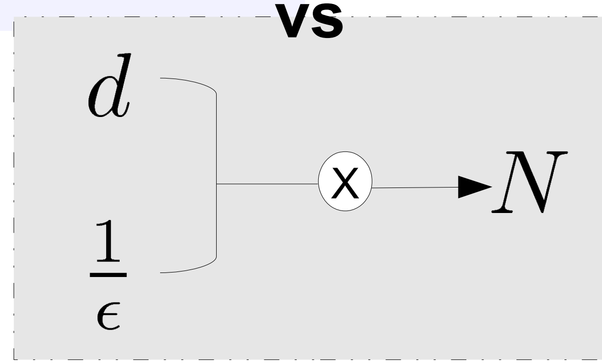
Take-home message #2

If $N_1 := 20 \cdot d$

$N_2 := 21 \cdot \frac{1}{\epsilon}$



vs



then

$$\mathbb{P}\{\ell_\delta(\gamma_F^*) > h_F^*\} \leq \epsilon$$

Increasing robustness

If $N_1 := 20 \cdot d$

$N_2 := 21 \cdot \frac{1}{\epsilon}$

then

$$\mathbb{P}\{\ell_\delta(\gamma_F^*) > h_F^*\} \leq \epsilon$$

Example: LMI

$$\min_{\gamma \in \mathbb{R}^{200}} \sum_{j=1}^{200} \gamma_j$$

$$\text{subject to:} \;\; \sum_{j=1}^{200} R_j(\delta) B(\delta) R_j(\delta)^T \gamma_j \preceq I$$

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \quad B(\delta) = \begin{pmatrix} \delta_1 & \delta_2 \\ \delta_2 & \delta_3 \end{pmatrix}; \quad R_j(\delta) = \begin{pmatrix} \cos\left(2\pi \frac{j-1}{T(\delta)}\right) & -\sin\left(2\pi \frac{j-1}{T(\delta)}\right) \\ \sin\left(2\pi \frac{j-1}{T(\delta)}\right) & \cos\left(2\pi \frac{j-1}{T(\delta)}\right) \end{pmatrix},$$

for $j = 1, \ldots, 200$, and $T(\delta) = 200 + 200^{2\delta_4}$.

Example: LMI

$\epsilon = 0.01$

Example: LMI

$\epsilon = 0.01$

Scenario Approach:

$\mathrm{N} = 29631$

Example: LMI

$\epsilon = 0.01$

Scenario Approach: N=29631

FAST

$N_1 = 4000$

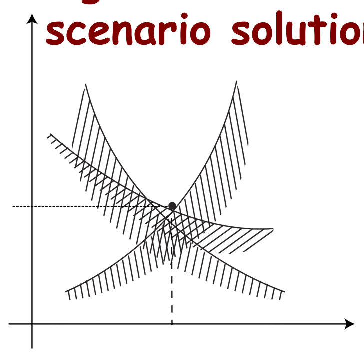
$N_2 = 2062$

$\oplus$ → N=6062

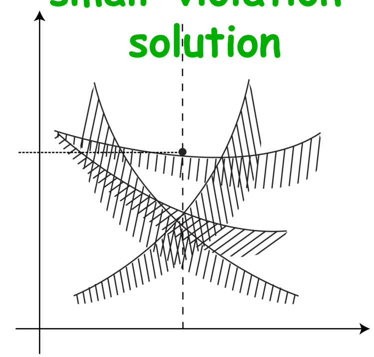
FAST algorithm

$\delta^{(1)}, \ldots, \delta^{(N_1)}$ → **Convex program** → *Intermediate solution* → **Updating** (with $\delta^{(N_1+1)}, \ldots, \delta^{(N_1+N_2)}$) → *FAST solution*

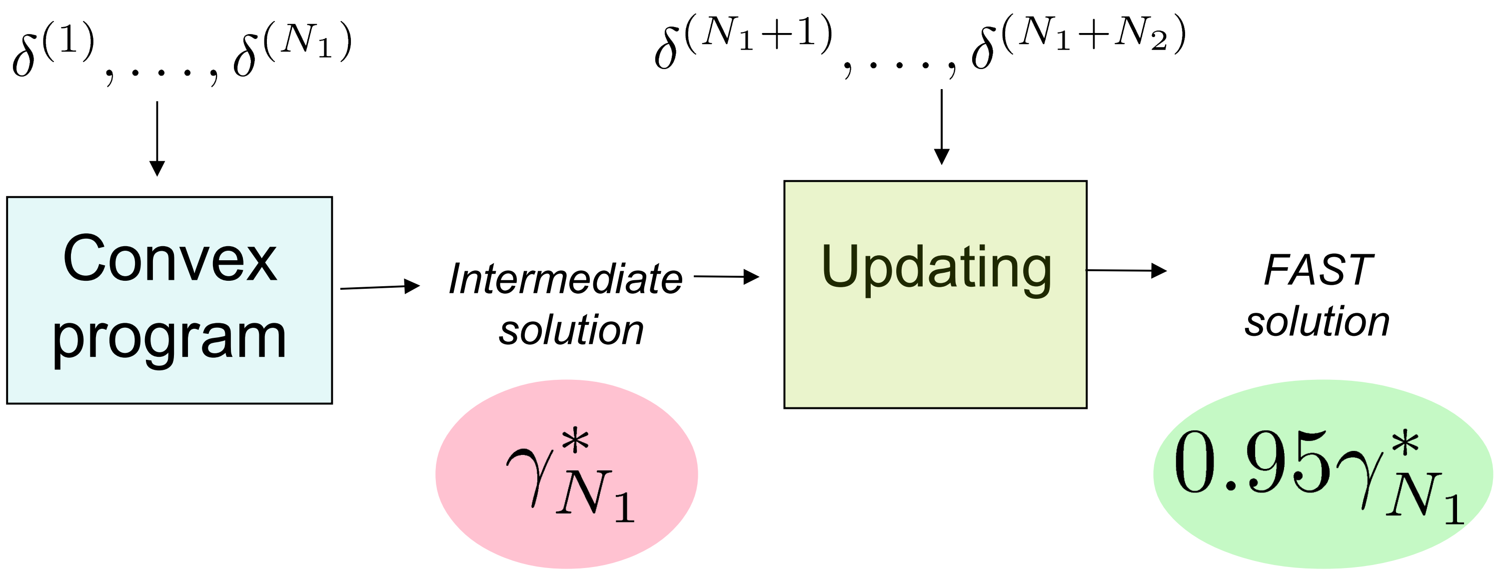
high-violation scenario solution



small-violation solution

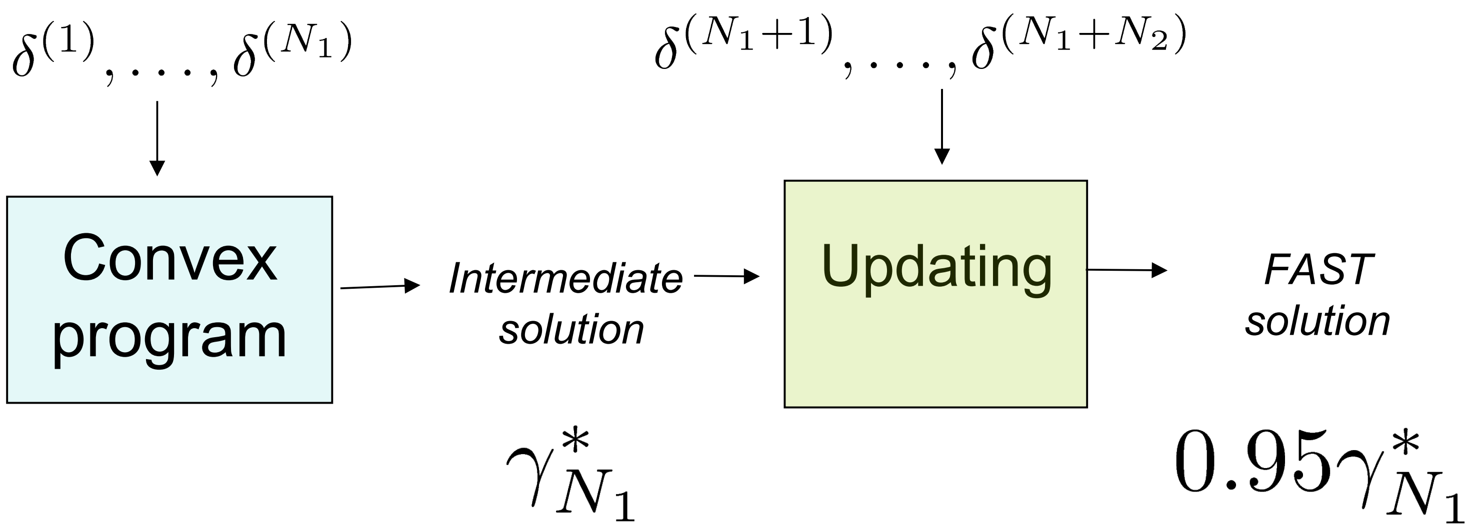


Example: LMI



Performance: -1.076 -1.024

Example: LMI



Performance: -1.076 -1.024

THANK YOU

The Scenario Approach

An issue

FAST gets around the issue

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Any questions?